

In the figure of the control loop above, we want to find the transfer function, so we solve for:

$$H[z] = \frac{\Phi_0[z]}{\Phi_I[z]}$$

In the time domain representation, we have:

$$\phi_{O}[n] = \phi_{O}[n-1] + \alpha \big[\phi_{I}[n] - \phi_{O}[n] \big] + \sum_{i=0}^{n} \beta \big[\phi_{I}[n-i] - \phi_{O}[n-i] \big]$$

The last integral is a first-order filter, which is easily shown to be represented in the z-domain as:

$$\frac{z}{z-1}$$

So using the bilinear z-transform, we convert the discrete time series to:

$$\Phi_0[z] = \Phi_0[z]z^{-1} + \alpha \left[\Phi_I[z] - \Phi_0[z] \right] + \beta \left[\Phi_I[z] - \Phi_0[z] \right] \frac{z}{z - 1}$$

Rearranging, weget:

$$\Phi_0[z]\left[1-z^{-1}+\alpha+\beta\frac{z}{z-1}\right] = \Phi_I[z]\left[\alpha+\beta\frac{z}{z-1}\right]$$

Multiplying both sides by $\frac{z}{z-1}$ gives us:

$$\Phi_0[z] \left[\frac{z-1}{z-1} + \left[\alpha + \beta \frac{z}{z-1} \right] \frac{z}{z-1} \right] = \Phi_I[z] \left[\alpha + \beta \frac{z}{z-1} \right] \frac{z}{z-1}$$
$$\Phi_0[z] \left[1 + \left[\alpha + \beta \frac{z}{z-1} \right] \frac{z}{z-1} \right] = \Phi_I[z] \left[\alpha + \beta \frac{z}{z-1} \right] \frac{z}{z-1}$$

Therefore,

$$H[z] = \frac{\Phi_0[z]}{\Phi_I[z]} = \frac{\left[\alpha + \beta \frac{z}{z-1}\right] \frac{z}{z-1}}{1 + \left[\alpha + \beta \frac{z}{z-1}\right] \frac{z}{z-1}}$$

We want to reformat this equation into the classical 2nd loop function:

$$H_{REF}[s] = \frac{\varpi_n^2}{s^2 + 2\zeta \varpi_n s + \varpi_n^2}$$

Using Tustin's method to move $H_{REF}[s]$ to the z-domain:

$$s = \frac{2}{T_s} \frac{\mathrm{Z} - 1}{\mathrm{Z} + 1}$$

If we work through the algebra, and substitute $\theta_n = \frac{\varpi_n T_s}{2}$, where θ_n is the undamped natural frequency, we get:

$$H_{REF}[z] = \frac{4\theta_n(\zeta + \theta_n)}{1 + 2\zeta\theta_n + \theta_n^2} \frac{z - \frac{\zeta}{\zeta + \theta_n}}{z^2 - 2\frac{1 + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2}z + \frac{1 - 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2}}$$

This looks messy now, but we can find substitutions for this equation in terms of our loop gains, α and β , where α is known as the proportional gain and β is known as the integral gain. Specifically,

$$\frac{1+\theta_n^2}{1+2\zeta\theta_n+\theta_n^2} = 1 - \frac{\alpha+\beta}{2}$$
$$\frac{\alpha+\beta}{2} = 1 - \frac{1+\theta_n^2}{1+2\zeta\theta_n+\theta_n^2} = \frac{1+2\zeta\theta_n+\theta_n^2}{1+2\zeta\theta_n+\theta_n^2} - \frac{1+\theta_n^2}{1+2\zeta\theta_n+\theta_n^2} = \frac{2\zeta\theta_n+2\theta_n^2}{1+2\zeta\theta_n+\theta_n^2}$$
$$\alpha+\beta = \frac{4\zeta\theta_n+4\theta_n^2}{1+2\zeta\theta_n+\theta_n^2}$$

Similarly,

$$\frac{1 - 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} = 1 - \alpha$$

$$\begin{aligned} \alpha &= 1 - \frac{1 - 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} = \frac{1 + 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} - \frac{1 - 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} = \\ \alpha &= \frac{4\zeta\theta_n}{1 + 2\zeta\theta_n + \theta_n^2} \end{aligned}$$

Which leaves,

$$\alpha = \frac{4\zeta\theta_n}{1+2\zeta\theta_n+\theta_n^2}$$
$$\beta = \frac{4\theta_n^2}{1+2\zeta\theta_n+\theta_n^2}$$
$$\theta_n = \frac{\varpi_n T_s}{2}$$