The Dodgy Dynamics of Economics

> "Maybe there is in human nature a deep-seated perverse pleasure in adopting and defending a wholly counterintuitive doctrine that leaves the uninitiated peasant wondering what planet he or she is on." (Solow 2006)

Today both Neoclassical and Post Keynesian economics proclaim that they are dynamic. Neoclassical macroeconomists embed the word "Dynamic" in the very title of their dominant class of models (DSGE, "Dynamic Stochastic General Equilibrium"), while Post Keynesian macroeconomists follow the lead of Wynne Godley (Godley 1999, Godley 2004, Godley and Lavoie 2005, Godley and Lavoie 2007) in devising stock-flow consistent models of financial flows between social entities (SFCA, the "Stock-Flow Consistent Approach"). This monetary, balance sheet approach to economics is necessarily a dynamic one.

However, a practitioner of dynamic methods from any other discipline would find what these two schools do when they "do dynamics" to be very peculiar. For a historian of economic thought, an equally bizarre peculiarity is that the method each school uses is to some degree in conflict with the underlying paradigm of that school.

Modern dynamic analysis

The defining differences between modern dynamics and dynamics as practised in the 19th century, when the fathers of Neoclassical economics adopted static methods because dynamics was regarded as too difficult,[1] are the shift from reductionism to holism, the importance of nonlinearity, the recognition of emergent properties in large scale systems, the rise of complex systems analysis, and the development of methods to enable such nonlinear, large scale, complex systems to be analysed by predominantly numerical methods.

Though the intertwined limits to reductionism, need for holism, and significance of nonlinear relations were first formalised by Poincare in the 19th century,[2] the pivotal demonstration of their significance came with Lorenz's demonstration of complex dynamics in a simple model of convection in 1967 (Lorenz 1963). He was critical of both approaches

---

[1] "We must carefully distinguish at the same time between the Statics and Dynamics of this subject. *The real condition of industry is one of perpetual motion and change*. Commodities are continually being manufactured and exchanged and consumed. If we wished to have a complete solution of the problem in all its natural complexity we should have to treat it as a problem of dynamics. But it would surely be absurd to attempt the more difficult question when the more easy one is yet so imperfectly within our power." Jevons, W. S. (1888). The Theory of Political Economy. Internet, Library of Economics and Liberty..

[2] This conception [of reductionism] was not without grandeur; it was seductive, and many among us have not finally renounced it; they know that one will attain the ultimate elements of things only by patiently disentangling the complicated skein that our senses give us; that it is necessary to advance step by step, neglecting no intermediary; that our fathers were wrong in wishing to skip stations; but they believe that when one shall have arrived at these ultimate elements, there again will be found the majestic simplicity of celestial mechanics. Poincaré, H. (1905, 1956). "Principles of Mathematical Physics." The Scientific Monthly **82**(4): 165-175.

that applied a conservation rule when the weather was clearly a dissipative system,[3] and linear approaches to weather modelling.[4]

His highly simplified model had just 3 equations and 3 parameters:

$$\begin{aligned}\frac{dx}{dt}&=(y-x)\times\sigma\\ \frac{dy}{dt}&=(\rho-z)\times x-y\\ \frac{dz}{dt}&=y\times x-\beta\times z\end{aligned}\qquad (1.1)^5$$

This simplicity resulted an unexpected level of complex behaviour. The system had not one equilibrium, but three:

$$\begin{bmatrix}x_{e1}\\ y_{e1}\\ z_{e1}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}x_{e2}\\ y_{e2}\\ z_{e2}\end{bmatrix}=\begin{bmatrix}\sqrt{\beta\times(\rho-1)}\\ \sqrt{\beta\times(\rho-1)}\\ \rho-1\end{bmatrix},\begin{bmatrix}x_{e3}\\ y_{e3}\\ z_{e3}\end{bmatrix}=\begin{bmatrix}-\sqrt{\beta\times(\rho-1)}\\ -\sqrt{\beta\times(\rho-1)}\\ \rho-1\end{bmatrix}\qquad (1.2)$$

For realistic parameter values, all 3 equilibria were unstable: the zero equilibrium was stable along two dimensions, but unstable on the third—a characteristic known as a *saddle node repeller*. The other two equilibria were stable along one dimension, but marginally unstable and cyclical along the other two. Far from being characterized by its equilibrium conditions, the equilibria in this model are places the system would never be, and the cycles generated by the model were aperiodic—in any given time path the values of the three system states never repeated.

The nonlinear instability of this system meant that, if out of equilibrium, the system would be attracted towards one of the equilibria only to be repelled from it, and ultimately flung back towards another equilibria—and repelled again. A clear pattern developed over time—as is evident from **Error! Reference source not found.**—but as a deterministic dynamic system, the numerical values of the 3 variables never recurred.

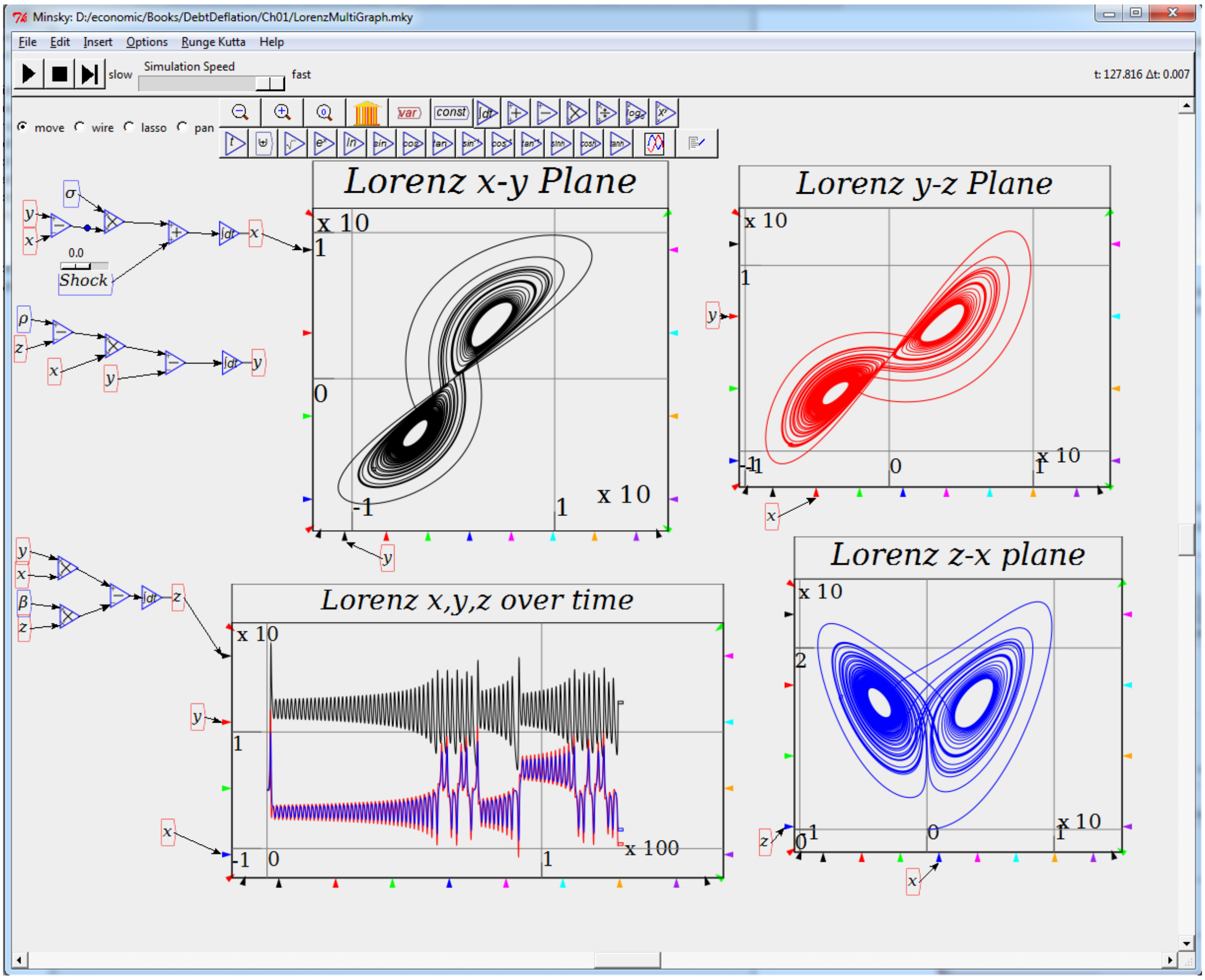
---

[3] "For certain purposes many systems may be treated as conservative systems, in which the total energy, or some other quantity, does not vary with time. In seeking the ultimate behaviour of a system, the use of conservative equations is unsatisfactory, since the ultimate value of any conservative quantity would then have to equal the arbitrarily chosen initial value." Lorenz, E. N. (1963). "Deterministic Nonperiodic Flow." Journal of the Atmospheric Sciences **20**(2): 130-141. (p. 131)

[4] "Indeed, in dissipative systems governed by a finite set of *linear* equations, a constant forcing leads to a constant response, while a periodic forcing leads to a periodic response. Hence, nonperiodic flow has sometimes been regarded as the result of nonperiodic or random forcing. The reasoning leading to these conclusions is not applicable when the governing equations are nonlinear." Ibid.

[5] "In these equations X is proportional to the intensity of the convective motion, while Y is proportional to the temperature difference between the ascending and descending currents, similar signs of X and Y denoting that warm fluid is rising and cold fluid is descending. The variable Z is proportional to the distortion of the vertical temperature profile from linearity, a positive value indicating that the strongest gradients occur near the boundaries". Ibid. (p. 135). The parameters (σ,ρ,β) represent respectively the ratio of the fluid's viscosity to its conductivity, the temperature difference from the bottom to the top of the box, and the ratio of the box's width to its height. Lorenz's default values were 10, 28 and 8/3.

Figure 1: Lorenz's model with the initial condition for (x,y,z) of (0.1,0,0) rather than the equilibrium value of (0,0,0)

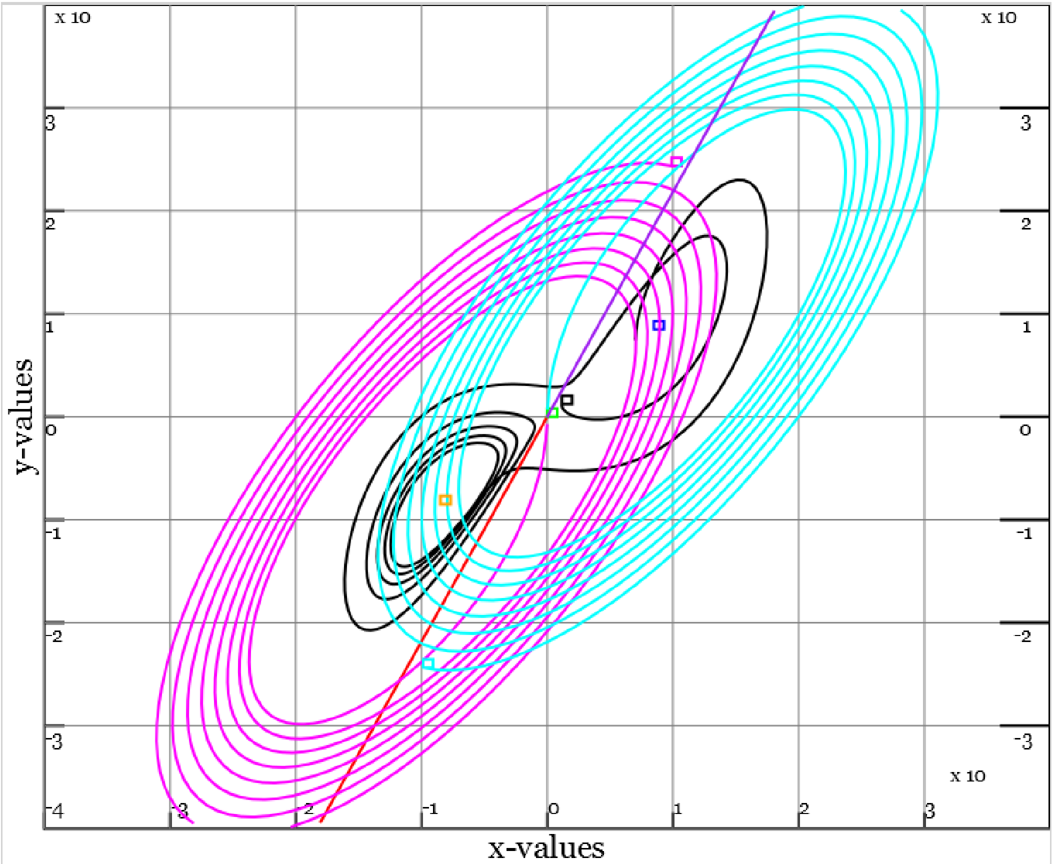


Lorenz christened this phenomenon "Deterministic Non-periodic Flow" (Lorenz 1963), while the term "strange attractors" was coined to describe these equilibria, and the behaviour of such a system was initially described as "chaotic behaviour", and later as "complex behaviour". This had been observed before in models of engineered systems,[6] but such systems could be designed to constrain the system to stay within bounds where a linear model was a reliable approximation.

No such option existed for the weather, and therefore the system's behaviour could not be approximated by a linear model in the vicinity of any one of these equilibria, as the linearizations of Lorenz's model in Figure 2 illustrate.

[6] Notably the Van der Pol oscillator, which describes endogenous oscillations in electric circuits.

Figure 2: The actual Lorenz system in black versus linear approximations around each of its 3 equilibria



Though the linear representation would accurately describe the system's behaviour in the immediate vicinity of the equilibrium, since these were unstable, the system would necessarily diverge from this vicinity, and the nonlinear terms would then dominate its behaviour. This meant that a common practice in science till that time—of linearizing a nonlinear model around its equilibrium, and then analyzing this tractable system—could not be applied to the weather, and any other system where engineering could not be relied upon to keep the system within linear bounds.

The system also displayed what became known as "sensitive dependence on initial conditions", in that slight errors between the actual state of the system and estimates of it used in the model were amplified exponentially over time. This phenomenon meant that the dream of truly long-term weather prediction had to be abandoned, since—even if forecasters possessed a completely accurate model of the weather—infinite accuracy of the measurement of current weather conditions was needed to prevent the complete loss of accuracy of future forecasts within a distinctly finite period. As Lorenz put it:

> When our results concerning the instability of non-periodic flow are applied to the atmosphere, they indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly. In view of the inevitably inaccuracy and incompleteness of weather observations, precise very-long-range forecasting would seem to be non-existent. (Lorenz 1963, p. 141)

Subsequent work by mathematicians established that pre-requisites for such aperiodic cyclical behaviour were firstly the dimensionality of the system—three or more nonlinear differential equations were required (Li and Yorke's celebrated "Period Three Implies Chaos" (Li and Yorke 1975)) and "mixing" of trajectories, so that even though two arbitrarily close trajectories diverged from each other:

> the interesting dynamics is confined to a globally finite region of the state space and thus the separated trajectories are necessarily folded back and can re-approach each other arbitrarily closely, infinitely many times. (Cvitanovic, Artuso et al. 2014, p. 8)

This and related discoveries showed that there were limits to reductionism—the process of trying to understand a high-order system by breaking it down into its constituent parts—and undermined the related conceit that Physics Nobel Laureate Philip Anderson described as "constructionism" in his seminal paper "More is Different" (Anderson 1972):

> the reductionist hypothesis does not by any means imply a "constructionist" one: The ability to reduce everything to simple fundamental laws does not imply the ability to start from those laws and reconstruct the universe…
>
> The constructionist hypothesis breaks down when confronted with the twin difficulties of scale and complexity. The behavior of large and complex aggregates of elementary particles, it turns out, is not to be understood in terms of a simple extrapolation of the properties of a few particles. Instead, at each level of complexity entirely new properties appear, and the understanding of the new behaviors requires research which I think is as fundamental in its nature as any other.
>
> That is, it seems to me that one may array the sciences roughly linearly in a hierarchy, according to the idea: The elementary entities of science X obey the laws of science Y

X	Y
Solid state or many-body physics	Elementary particle physics
Chemistry	Many-body physics
Molecular biology	Chemistry
Cell biology	Molecular biology
…	…
Psychology	Physiology
Social sciences	Psychology

> But this hierarchy does not imply that science X is "just applied Y". At each stage entirely new laws, concepts, and generalizations are necessary, requiring inspiration and creativity to just as great a degree as in the previous one. Psychology is not applied biology, nor is biology applied chemistry. (Anderson 1972, p. 393)

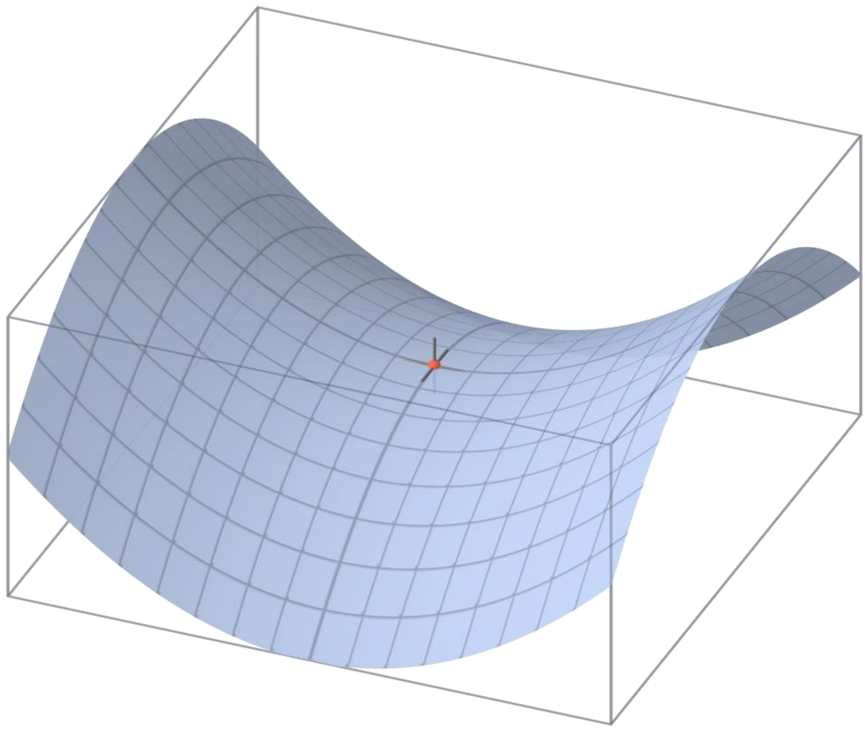
Though there are certainly some economists who are aware of these developments and apply them in their own research (Flaschel and Semmler 1986, Goodwin 1990, Franke and Lux 1993, Delli Gatti, Gallegati et al. 1998, Chiarella and Flaschel 2000, Lux 2001, Asada, Chen et al. 2006, Sushko, Wegener et al. 2009), the dominant practices in both Neoclassical and Post Keynesian macroeconomics ignore these advances in the sciences.

A crack in the paradigm: Neoclassical Macroeconomics

Micro foundations and an equilibrium methodology are the heart of the Neoclassical paradigm (Varoufakis and Arnsperger 2006). DSGE models are clearly micro-founded—though as is usual in that school, crucial problems with aggregation (in this case the Sonnenschein-Mantel-Debreu theorem (Sonnenschein 1972, Kirman 1989)) are ignored—and the method is that peculiar Neoclassical oxymoron of equilibrium dynamics. But the core model on which DSGE models are based—the Ramsey growth model (Ramsey 1928)—is mathematically an undeniably unstable one.

In the past, Neoclassical economists have been at great pains to either remove instability from economic models—as Hicks did with his "trade cycle" reworking of Harrod's unstable growth model (Harrod 1939, Harrod 1948, Hicks 1949)—or they have ignored essentially unstable aspects of their models (as General Equilibrium modelers did in the 1960s-1980s when "general equilibrium" meant devising multi-commodity input-output computable models of production and exchange, and the "dual instability" property of Leontief technology was ignored (Jorgenson 1960, Jorgenson 1961, Jorgenson 1963, McManus 1963, Blatt 1983)).

Derived from the fiction of a classless, harmonious (or benevolently ruled) eternal society, the Ramsey growth model is expressed in two equations that generate what mathematicians call a "Saddle Point" equilibrium—which is unstable in the same sense that a saddle is an unstable object on which to rest a ball (see Fi



gure 3).

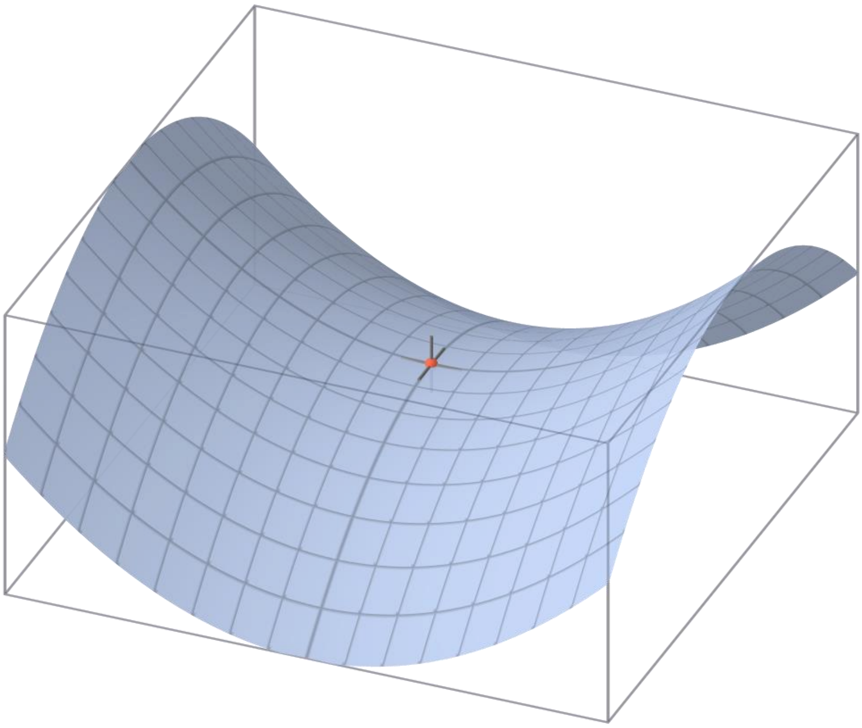


Figure 3: The shape of the function f(x,y) = x2-y2 and its saddle point equilibrium

This instability was an accidental by-product of Ramsey's endeavour, which was to answer the question "how much of its income should a nation save?" within the context of a Neoclassical vision of the world. Ramsey argued that the following abstractions, though extreme, were necessary for the task at hand:[7]

> we have to suppose that our community goes on for ever without changing … in its capacity for enjoyment or in its aversion to labour; that enjoyments and sacrifices at different times can be calculated independently and added; … We also ignore altogether distributional considerations, … we neglect the differences between different kinds of goods and different kinds of labour, … so that we can speak simply of quantities of capital, consumption and labour without discussing their particular forms. (Ramsey 1928, p. 543)

With these abstractions, Ramsey derived a rule for the rate of consumption from now till when the point of "Bliss" would be attained—thus specifying not only an ultimate consumption objective, but today's as well, and all points in between. The model reduces to two nonlinear differential equations in the capital to labor intensity *k* and the per capita consumption rate *c*:

[7] I omit those assumptions which have themselves been dropped by modern Neoclassical macroeconomics

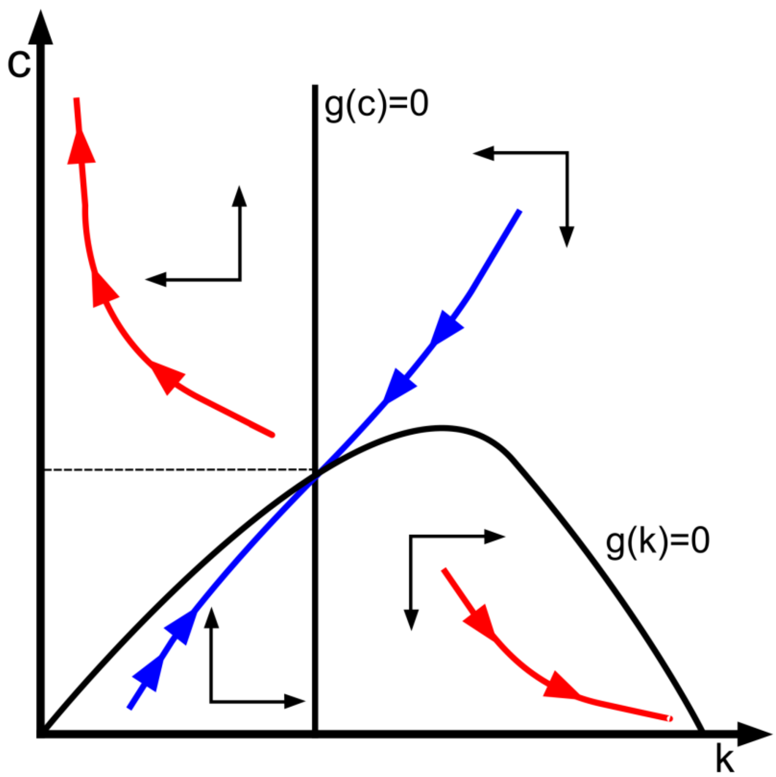
$$\frac{d}{dt}k = f(k) - c - (n+g+\delta)\cdot k$$
$$\frac{d}{dt}c = \frac{c}{\rho}\cdot\left(\left(\frac{d}{dk}f(k)\right) - (\delta + \rho\cdot g + n - \upsilon)\right) \tag{1.3}$$
[8]

With a Cobb-Douglas production function used for f(k), this model is

$$\frac{d}{dt}k = k^{\alpha} - c - (n+g+\delta)\cdot k$$
$$\frac{d}{dt}c = \frac{c}{\rho}\cdot\left(\alpha\cdot k^{\alpha-1} - (\delta+\rho\cdot g+n-\upsilon)\right) \tag{1.4}$$

Prior to Ramsey's work, a Neoclassical prayer would have been that the non-trivial equilibrium of this system would prove to be stable. Unfortunately that did not prove to be: the system is unstable around its equilibrium, as is normally illustrated by a phase diagram like that in Figure 4.

Figure 4: Wikipedia illustration of a phase diagram for the Ramsey growth model showing its instability



[8] Here $k = \frac{K}{A\cdot L}$ where K is the capital stock, A is labor productivity and L labor, $c = \frac{C}{A\cdot L}$ where C is consumption, *n*, *g* and δ are the rates of population growth (exponential growth rate of *L*), technical progress (the exponential growth rate of *A*) and depreciation respectively, *f(k)* is the production function (normally a Cobb-Douglas in which $f(k) = k^{\alpha}$), ρ is the coefficient in a constant risk aversion utility function $u(c) = \frac{c^{1-\rho}}{1-\rho}$, and υ is the rate of time discount.

The instability of this path was unproblematic for Ramsey, since he was specifying a social ideal—in the absence of many of the factors that in fact define society—rather than a hypothesized actual path of the economy. But when this was chosen by the leaders of the "Real Business Cycle" Revolution as the basis for their micro-founded replacement for IS-LM macroeconomics, its consistency with the desired paradigmatic emphasis upon microfoundations had the undesirable sibling of fundamental instability.

This instability could not be ignored—a saddle is a saddle—so instead it was made consistent with the paradigm-defining focus upon equilibrium by introducing the concept of a "jump variable" to dynamics, and by extending Muth's concept of "Rational Expectations" from an Agent with the capacity to predict the future of a single market to one with the capacity to understand and therefore predict the dynamics of the entire macroeconomy for its indefinite future. Endowed with both complete knowledge of the present, and the capacity to predict the future, this Agent was thus be able to "jump" current consumption so as to land on the sole unstable "saddle path" to the saddle point equilibrium.

To paraphrase Douglas Adams from *The Hitchhiker's Guide to the Galaxy*, this definition "is obviously some strange usage of the word 'rational' that I wasn't previously aware of". Calling the capacity to predict the future "rational" was a deliberate distortion of the ordinary meaning of the word rational, as was conceded—or more accurately bragged about—by Barro in 1984:

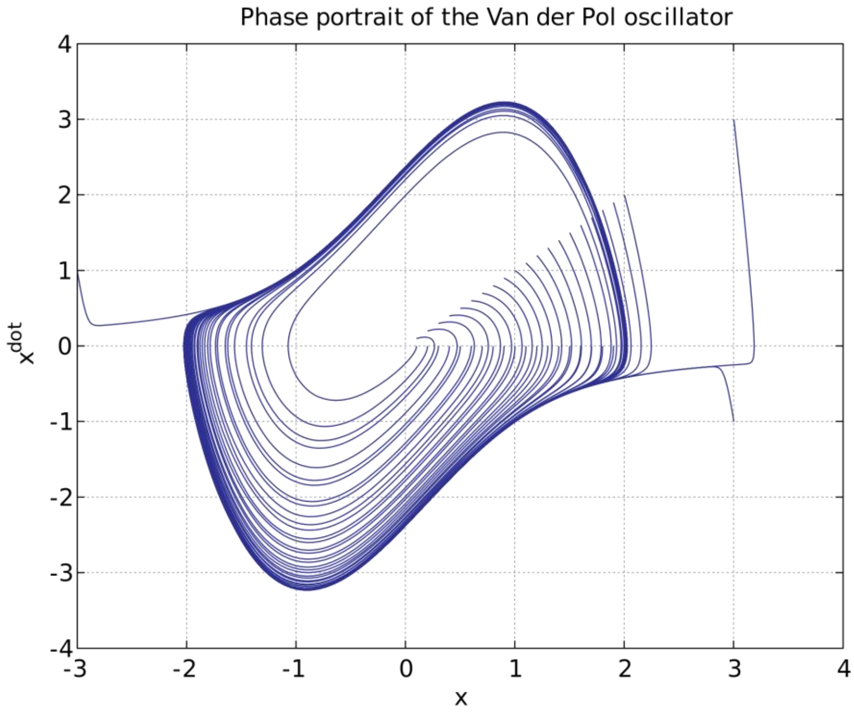
> One of the cleverest features of the rational expectations revolution was the appropriation of the term "rational." Thereby, the opponents of this approach were forced into the defensive position of either being irrational or of modeling others as irrational, neither of which are comfortable positions for most economists." (Barro 1984)

Rather than taking "the defensive position", I suggest that critics instead adopt the attacking position of describing this concept—the capacity to foretell the whole future of the economy—as a dictionary would, as Prophetic Expectations.

Endowed with Prophetic Expectations, and being the only consumer in the economy, the Prophetic Agent is thus able in DSGE models to instantly (for New Classicals) or gradually (for New Keynesians) modify the existing rate of consumption in the light of shocks to tastes or technology so that the economy "jumps" from its current disturbed-from-equilibrium state to a new equilibrium-consistent state.

This concept of a "jump variable" is another departure from standard mathematical dynamics. In standard mathematical dynamics, a system of differential equations defines a whole spectrum of non-intersecting time paths in a phase space—as Figure 5 illustrates for the Van der Pol oscillator.

Figure 5: Wikipedia illustration of a phase space



But in Prophetic Expectations Macroeconomic models, the "jump variable" moves from one non-intersecting time path to another, under the influence of the Prophetic Agent's knowledge of the entire dynamics of the model economy in which he is embedded. Thus if the model economy was disturbed by a stochastic shock to the "deep parameters" of either preferences or technology, the Prophetic Agent would find himself not on the unstable path towards the saddle point equilibrium defined by the new parameter values, and would "jump" to that new unstable path by changing today's consumption in the light of the new information about the new location of the saddle point equilibrium in the distant future.

In the original Real Business Cycle models, this meant that the Prophetic Agent—who represented the entire economy under the fiction of the Representative Agent—was always in equilibrium. So even events as severe as the Great Depression were described not as disequilibrium events, nor even as unsatisfactory ones, but as optimal adjustments to some changed set of circumstances. This is how one of the originators of Real Business Cycle models, Edward Prescott, described the Great Depression:

> From the perspective of growth theory, the Great Depression is a great decline in steady-state market hours. I think this great decline was the unintended consequence of labor market institutions and industrial policies designed to improve the performance of the economy. Exactly what changes in market institutions and industrial policies gave rise to the large decline in normal market hours is not clear. But, then, neither is it clear why market hours are so low in France and Spain today.
>
> The Marxian view is that capitalistic economies are inherently unstable and that excessive accumulation of capital will lead to

> increasingly severe economic crises. Growth theory, which has proved to be empirically successful, says this is not true. The capitalistic economy is stable, and absent some change in technology or the rules of the economic game, the economy converges to a constant growth path with the standard of living doubling every 40 years. In the 1930s, there was an important change in the rules of the economic game. This change lowered the steady-state market hours. The Keynesians had it all wrong. In the Great Depression, employment was not low because investment was low. Employment and investment were low because labor market institutions and industrial policies changed in a way that lowered normal employment. (Prescott 1999, p. 3)

This perspective was (and is) too much for Neoclassicals with a somewhat closer grip on reality, which led to the development of New Keynesian DSGE models in which frictions and imperfections slowed down the Prophetic Agent's rate of return to the unstable saddle path. Hence periods of recessions were indeed sub-optimal (as indeed was the Great Depression), but ultimately the Prophetic Agent and the economy He[9] represented would return to the saddle path.

In practice, DSGE models are generated by incredibly complicated derivation from aggregation-problem-free microeconomic first principles, and lead to a system of unstable linear first order difference equations, which are derived by linearizing a nonlinear system around its unstable equilibrium. To apply these analytically insoluble models requires an equally complicated numerical procedure.

Both stages of complication—rather than complexity—are intellectually highly demanding, and appeal to the inner nerd that any mathematically-inclined theorist must have. But seen from the perspective of a genuine science like meteorology, they are also quite bizarre.

The linearization of a model around its unstable equilibrium directly violates a fundamental principle of modern dynamic analysis, as outlined earlier.

The very idea of a "saddle path" violates the principle that a saddle point equilibrium is unstable, as does the introduction of a "jump variable" to enable the model to shift from one dynamic trajectory to another. Both deviations from standard dynamic analysis are justified as means of taking the capacity of humans to make conscious decisions into account—the so-called "rationality" defence. But two fundamental objections can be made to this defence.

The first is that this is a defence not of the assumption of rationality, but a defence of the assumption of Omnipotence. The capacity to predict the future of the basis of an accurate model of reality is not rationality but the capacity for accurate prophecy—which is why I believe critics should describe this approach as Prophetic Expectations Macroeconomics, and reclaim the word rationality on behalf of the English language.

The second is more methodological in nature. "Rational expectations" is described by its promoters as meaning "model-consistent expectations"—that the agent's expectations are consistent with the model in which He is embedded. But the model ***is*** a model: it is not

[9] Such a being is surely a God rather than a mortal, and therefore deserves a capitalised pronoun.

reality, whose existence independent of their models is something I hope Prophetic Expectations macroeconomists would concede. So if a Prophetic Agent does actually exist, He will reside in the actual economy, and not in any model of it. Call this agent the Reality Consistent Prophetic Agent, or RCPA. On the other hand, the Prophetic Agent in any given DSGE model is the "Model Consistent Prophetic Agent", or MCPA.

Since there is a plethora of DSGE models, each with their own MCPA, there is a multitude of MCPAs that are inconsistent with both each other and the hypothetical RCPA. The behaviour of the actual economy therefore cannot be used as a guide to calibration of the "deep parameters" of a given DSGE model's MCPA. Indeed, were the actual "deep parameters" for the hypothesized RCPA injected into any DSGE model, the expectations of this RCPA would necessarily differ from the behaviour of the model itself, so that the "jump variable" would not jump onto the model's stable path, and endogenous cyclical fluctuations would result, rather than convergence to the unstable "saddle path". The concept of "model-consistent expectations" is thus itself irrational (in the English-language sense of the word).

A crack in the paradigm: Post Keynesian Macroeconomics

The Post Keynesian School's mathematical dynamic modeling of the economy can be traced back to Kalecki (Kalecki 1935, Kalecki 1936, Kalecki 1937) and Harrod (Harrod 1936, Harrod 1939, Harrod 1948) in the mid to late 1930s.

Though Kalecki's initial model abstracted from growth, his analysis of cycles argued that non-equilibrium dynamics were inherent even in the "marginal efficiency of investment" explanation that Keynes provided in the General Theory (which Keynes later abandoned for the argument that the desire to investment and produce capital good was motivated by "the relation between their costs of production and the prices which they are expected to realize in the market" (Keynes 1937, p. 217)):

> Let us assume that the rate of investment has really, say, risen so much that the new level of investment prices and the initial state of expectations give a marginal efficiency equal to the rate of interest. The increase of investment, however, will cause not only the prices of investment goods to rise, but also a rise of prices … and employment in all branches of trade. Thus, because "the facts of the existing situation enter, in a sense disproportionately, into the formation of our long-term expectations," the state of expectations will improve and the marginal efficiency of assets appears again higher than the rate of interest. Consequently "equilibrium" is not reached and the investment continues to rise.
>
> We see now that the Keynesian conception, which tells only how great investment will be if the given "disequilibrium" changes into an "equilibrium," encounters a difficulty in this respect also, for it appears that the rise of investment does not lead to "equilibrium " at all (in any case, not to immediate "equilibrium "). (Kalecki 1937, p. 84).

Harrod's growth model emphasized the endogenous instability of the growth path, and he insisted that both growth and cycles had to be explained jointly:

> Attempts to construct a dynamic theory have recently been proceeding … by the study of time lags between certain adjustments… In these studies there is some doubt as to the nature of the trend on which the oscillation is superimposed… Moreover it is possible … that the trend of growth may itself generate forces making for oscillation… The study of the operation of the forces maintaining a trend of increase and the study of lags should go together. (Harrod 1939, pp. 14-15)

His model, derived by dynamizing the *General Theory* equality of *ex-post* savings and investment, and then comparing these to hypothesized *ex-ante* functions, generated the conclusion that what Harrod called the "warranted growth rate" (which was the rate that motivated capitalists to maintain the current rate of investment) was unstable:

> A unique warranted line of growth is determined... On either side of this line is a 'field' in which centrifugal forces operate... Departure from the warranted line sets up an inducement to depart further from it. The moving equilibrium of advance is thus a highly unstable one" (Harrod 1939, p. 23)

Kaldor subsequently emphasized the vital role of nonlinearity in explaining sustained endogenous cycles, by noting that if linear functions were used for *ex-ante* investment and savings, these would imply either "*more* stability than the real world, in fact, appears to possess" or that "the economic system would always be rushing either towards a state of hyper-inflation … or towards total collapse…" (Kaldor 1940, p. 80) He therefore reasoned that

> Since thus neither of these two assumptions can be justified, we are left with the conclusion that the *I* and *S* functions cannot both be linear." (Kaldor 1940, p. 81)

The great champion of endogenous cyclicality and nonlinear analysis in the Post Keynesian tradition was Richard Goodwin (Goodwin 1950, Goodwin 1967, Goodwin 1985, Goodwin 1986, Goodwin 1990, Goodwin 1990, Goodwin, Pacini et al. 1992, Goodwin 1993, Goodwin and Freeman 1996, Velupillai 1998), and his deep knowledge of mathematics enabled him to identify the weakness of Harrod's endeavour to explain both growth and cycles out of what was fundamentally a first order linear differential equation (Velupillai 1998, p. 1438).

> The simple linearized pendulum theory … fails to explain most observed oscillations, i.e., the ones that neither die away nor explode … self-maintaining cycles occur because of essential nonlinearities in the structure of the system. Since Professor Hicks proposes a theory which will explain the maintenance of oscillation, we can be sure, on formal grounds, that this implies a non-linearity. (Goodwin 1950, p. 317-18)

Endogenous cyclicality, nonlinearity, and the instability of growth in a capitalist economy, are also essential aspects of Minsky's Financial Instability Hypothesis:

> The natural starting place for analyzing the relation between debt and income is to take an economy with a cyclical past that is now doing well…
>
> Stable growth is inconsistent with the manner in which investment is determined in an economy in which debt-financed ownership of capital assets exists, and the extent to which such debt financing can be carried is market determined. It follows that the fundamental instability of a capitalist economy is upward. The tendency to transform doing well into a speculative investment boom is the basic instability in a capitalist economy. (Minsky 1982, p. 65)

These factors—endogenous cyclicality, the instability of growth, the co-determination of growth and cycles, and the importance of nonlinearity—can therefore be said to be part of the foundations of Post Keynesian macroeconomics. Yet none of these factors have played a necessary or even a substantial role in Stock Flow Consistent Approach (SFCA) modelling to date.

While the essentially monetary nature of SCFA models, the strict methodological attention to stock-flow consistency, and the sequential nature of SCFA models—in contrast to the Neoclassical emphasis upon equilibrium—distinguish it from the Neoclassical paradigm, in these respects SCFA models are more consistent with the Neoclassical paradigm than are DSGE models, with their inherent instability.

The standard SCFA practice proceeds from the rigorous expression of the flows between different sectors of the economy using double-entry bookkeeping (with an emphasis on the financial flows as opposed to physical flows), follows up with the statement of a set of behavioural relations to determine these flows, and concludes with the derivation of a set of stock-flow consistent equations (which requires the dropping of one equation for reasons of redundancy). The equilibrium of this system is then located using numerical methods, and the model is subjected to shocks from this equilibrium—to which it returns after delayed determined by the time delays built into the behavioural equations. As Caverzasi and Godin put it:

> The steps leading up to a numerical solution are the following: First, numerical values for the parameters are individuated, generally basing the choices on the observation of stylized facts. Second, the model is calibrated or estimated—that is, values for the different parameters and exogenous variables are determined. Once this is done, a steady state is usually computed. The third and last step consists of simulations. In practice, these are changes in the value of the parameters or of the exogenous variables, which allows us to see how the economy reacts. This methodology is, by far, the most widely used among SFC practitioners. (Caverzasi and Godin 2013, p. 8)

In the vast majority of SCFA models, this results in a set of linear first order difference equations, the parameters of which have been chosen with an eye to the stability of its equilibrium in those simulations, rather than to economic realism—as Caverzasi and Godin put it:

> Taylor (2008) points out the importance of these stock-flow norms and shows how sometimes non-realistic values have to be assumed for the sake of having a realistic steady state. (Caverzasi and Godin 2013, p. 8)

This situation need not be endemic to SFCA modelling: there is no methodological or philosophical reason why SFCA models fail to acknowledge the structural and behavioural nonlinearities in the economy, or produce equilibrium outcomes rather than far from equilibrium dynamics. In contrast to the situation with DSGE modelling, where the acknowledgement of nonlinearity and complex dynamics would challenge the underlying paradigm, the same factors could be introduced into SFCA models and make them ***more*** consistent with the underlying paradigm, not less. The failure to do so to date is more an unconscious result of the evolution of the SFCA in partial isolation from other strands in Post Keynesian economics—especially that represented by Goodwin with his emphasis upon nonlinearity and endogenous cycles—than a necessity or a deliberate choice.

The extent to which this outcome was unconscious can be seen in a critical overview of SCFA modelling by Kinsella (Kinsella 2011), in which he notes that:

> There may also be a problem of chaos and complexity within these models. Obviously sensitive dependence on initial conditions does not mean the models are intrinsically chaotic or capable of generating complex dynamics, but the recursive nature of the modeling, the existence of multiple feedbacks within each models and the computation issues I and my co-authors have come across when practically trying to model a real economy give me pause that there might be the seeds of a complex system somewhere within stock flow modelling… (Kinsella 2011, p. 7)

There are obvious reasons why chaos and complexity ***should*** reside within not merely SFCA models but the economy itself, with the simplest being that identified by Goodwin in his seminal model (Goodwin 1967): the wage rate and the level of employment are both economic variables, and the wage level which is a product of these two variables is a factor in determining the rate of profit, so that there is at least this inherent structural nonlinearity in the economy. Many other structural nonlinearities can be identified, as Chiarellia, Faschel et al. do in their research (Chiarella and Flaschel 1996, Chiarella and Flaschel 2000, Agliari, Chiarella et al. 2006, Chen, Chiarella et al. 2006, Chiarella, Flaschel et al. 2006, Asada, Chiarella et al. 2010, Chiarella, Flaschel et al. 2012, Flaschel, Greiner et al. 2012), in addition to the nonlinear investment reactions of firms to tranquil economic conditions that is an important aspect of Minsky's Hypothesis.

Thus while SCFA models are deliberately dynamic, their approach to dynamics can be characterized as casual rather than causal. This is apparent in the almost exclusive use of "discrete-time" or difference equations in SFCA models. This practice appears to have originated in Wynne Godley's role as an empirical macroeconomist dealing with quarterly

data, and his intuition that since the data was discrete, so too should be the modelling (Lavoie 2014).

This is a false intuition. Difference equations are appropriate where two conditions apply: the underlying phenomena are themselves discrete—as is the births of all life forms; and where all such actions by the agents being modelled are synchronised—as are the births of a minority of species, such as red crabs on Christmas Island. In all other cases, continuous time methods are a superior "top-down" rendition of the underlying model, because the asynchronous nature of the discrete events—for example, the birth of human babies—means that a continuous time equation more accurately captures the underlying process at the aggregate level.

It implies a level of synchronicity in the actions of economic agents which simply does not exist. All investment decisions are not made at once, but at many different times. A first order time lag captures that process more accurately than a one year (or quarter) time delay.

The use of simple first-order time delays (and therefore either yearly or quarterly time dependencies) for virtually all relationships, including consumption as well as investment, and often these lags are introduced because they are needed to avoid what are known as "algebraic loops", rather than because they fit the data—as Godley noted in an influential paper:

> I have introduced lags into the consumption function and into the asset-holding functions whenever simultaneous interdependence threatened to generate meaningless oscillations. (Godley 1999, p. 409)

The proper use of difference equations requires an accurate specification of the average time delay involved in a decision—which would result in consumption having a time delay in the order of weeks while investment would have a time delay in the order of quarters or years. Though this could be done, another problem with difference equation models then arises: the time delays become part of the structure of the model, rather than parameters of the model which can easily be varied, as is the case with differential equation systems. Dynamic adjustments requires re-writing the model in the case of difference equations, whereas time lags values can vary during a simulation in the case of differential equations.

The Future

References

Agliari, A., C. Chiarella and L. Gardini (2006). "A Re-evaluation of Adaptive Expectations in Light of Global Nonlinear Dynamic Analysis." Journal of Economic Behavior and Organization **60**(4): 526-552.

Anderson, P. W. (1972). "More Is Different." Science **177**(4047): 393-396.

Asada, T., P. Chen, C. Chiarella and P. Flaschel (2006). "Keynesian Dynamics and the Wage-Price Spiral: A Baseline Disequilibrium Model." Journal of Macroeconomics **28**(1): 90-130.

Asada, T., C. Chiarella, P. Flaschel, T. Mouakil, C. R. Proano and W. Semmler (2010). "Stabilizing an Unstable Economy: On the Choice of Proper Policy Measures." Economics: The Open-Access, Open-Assessment E-Journal **4**.
Barro, R. J. (1984). "Rational Expectations and Macroeconomics in 1984." The American Economic Review **74**(2): 179-182.
Blatt, J. M. (1983). Dynamic economic systems: a post-Keynesian approach. Armonk, N.Y, M.E. Sharpe.
Caverzasi, E. and A. Godin (2013). Stock-flow Consistent Modeling through the Ages. Working Paper Collection. Annandale-on-Hudson, Levy Economics Institute.
Chen, P., C. Chiarella, P. Flaschel and W. Semmler (2006). "The Feedback Channels in Macroeconomics: Analytical Foundations for Structural Econometric Model Building." Central European Journal of Operations Research **14**(3): 261-288.
Chiarella, C. and P. Flaschel (1996). "Real and Monetary Cycles in Models of Keynes-Wicksell Type." Journal of Economic Behavior and Organization **30**(3): 327-351.
Chiarella, C. and P. Flaschel (2000). The Dynamics of Keynesian Monetary Growth. Cambridge, Cambridge University Press.
Chiarella, C., P. Flaschel, F. Hartmann and C. R. Proano (2012). "Stock Market Booms, Endogenous Credit Creation and the Implications of Broad and Narrow Banking for Macroeconomic Stability." Journal of Economic Behavior and Organization **83**(3): 410-423.
Chiarella, C., P. Flaschel and H. Hung (2006). "Interacting Business Cycle Fluctuations: A Two-Country Model." Singapore Economic Review **51**(3): 365-394.
Cvitanovic, P., R. Artuso, R. Mainieri, G. Tanner and G. a. Vattay (2014). Chaos: Classical and Quantum.
Delli Gatti, D., M. Gallegati and D. Mignacca (1998). "Nonlinear Dynamics and European GNP Data." Studies in Nonlinear Dynamics and Econometrics **3**(1): 43-59.
Flaschel, P., A. Greiner, C. Logeay and C. Proano (2012). "Employment Cycles, Low Income Work and the Dynamic Impact of Wage Regulations: A Macro Perspective." Journal of Evolutionary Economics **22**(2): 235-250.
Flaschel, P. and W. Semmler (1986). The Dynamic Equalization of Profit Rates for Input-Output Models with Fixed Capital. Competition, instability, and nonlinear cycles: Proceedings of an internatonal conference, New School for Social Research, New York, USA, March 1985. W. Semmler. New York, Springer**:** 1-34.
Franke, R. and T. Lux (1993). "Adaptive Expectations and Perfect Foresight in a Nonlinear Metzlerian Model of the Inventory Cycle." Scandinavian Journal of Economics **95**(3): 355-363.
Godley, W. (1999). "Money and Credit in a Keynesian Model of Income Determination." Cambridge Journal of Economics **23**(4): 393-411.
Godley, W. (2004). "Money and Credit in a Keynesian Model of Income Determination: Corrigenda." Cambridge Journal of Economics **28**(3): 469-469.
Godley, W. and M. Lavoie (2005). "Comprehensive Accounting in Simple Open Economy Macroeconomics with Endogenous Sterilization or Flexible Exchange Rates." Journal of Post Keynesian Economics **28**(2): 241-276.
Godley, W. and M. Lavoie (2007). Monetary Economics: An Integrated Approach to Credit, Money, Income, Production and Wealth. New York, Palgrave Macmillan.
Goodwin, R. M. (1950). "A Non-Linear Theory of the Cycle." The Review of Economics and Statistics **32**(4): 316-320.

Goodwin, R. M. (1967). A growth cycle. Socialism, Capitalism and Economic Growth. C. H. Feinstein. Cambridge, Cambridge University Press**:** 54-58.
Goodwin, R. M. (1985). "A Personal Perspective on Mathematical Economics." Banca Nazionale del Lavoro Quarterly Review(152): 3-13.
Goodwin, R. M. (1986). "The Economy as an Evolutionary Pulsator." Journal of Economic Behavior and Organization **7**(4): 341-349.
Goodwin, R. M. (1990). Chaotic economic dynamics. Oxford, Oxford University Press.
Goodwin, R. M. (1990). "The Complex Dynamics of Innovation, Output, and Employment." Structural Change and Economic Dynamics **1**(1): 119-131.
Goodwin, R. M. (1993). My Erratic Progress toward Economic Dynamics: Remarks Made at Banquet, Tuesday, April 18, 1989. Nonlinear dynamics and evolutionary economics. Oxford, Oxford University Press**:** 303-306.
Goodwin, R. M. and C. Freeman (1996). The Economy as an Evolutionary Pulsator. Long wave theory. Cheltenham, Elgar**:** 158-165.
Goodwin, R. M., P. M. Pacini, A. Vercelli and N. Dimitri (1992). Nonlinear Economic Dynamics and Chaos: An Introduction. Macroeconomics: A survey of research strategies. Oxford, Oxford University Press**:** 236-291.
Harrod, R. F. (1936). The Trade Cycle: An Essay. Oxford, Clarendon Press.
Harrod, R. F. (1939). "An Essay in Dynamic Theory." The Economic Journal **49**(193): 14-33.
Harrod, R. F. (1948). Towards a Dynamic Economics. London, Macmillan.
Hicks, J. R. (1949). "Mr. Harrod's Dynamic Theory." Economica **16**(62): 106-121.
Jevons, W. S. (1888). The Theory of Political Economy. Internet, Library of Economics and Liberty.
Jorgenson, D. W. (1960). "A Dual Stability Theorem." Econometrica **28**(4): 892-899.
Jorgenson, D. W. (1961). "Stability of a Dynamic Input-Output System." The Review of Economic Studies **28**(2): 105-116.
Jorgenson, D. W. (1963). "Stability of a Dynamic Input-Output System: A Reply." The Review of Economic Studies **30**(2): 148-149.
Kaldor, N. (1940). "A Model of the Trade Cycle." The Economic Journal **50**(197): 78-92.
Kalecki, M. (1935). "A Macrodynamic Theory of Business Cycles." Econometrica **3**(3): 327-344.
Kalecki, M. (1936). "Comments on the Macrodynamic Theory of Business Cycles." Econometrica **4**(4): 356-360.
Kalecki, M. (1937). "A Theory of the Business Cycle." The Review of Economic Studies **4**(2): 77-97.
Keynes, J. M. (1937). "The General Theory of Employment." The Quarterly Journal of Economics **51**(2): 209-223.
Kinsella, S. (2011). Words to the Wise: Stock Flow Consistent Modeling of Financial Instability, Geary Institute, University College Dublin, Working Papers: 201130**:** 14 pages.
Kirman, A. (1989). "The Intrinsic Limits of Modern Economic Theory: The Emperor Has No Clothes." Economic Journal **99**(395): 126-139.
Lavoie, M. (2014). SFC modeling isn't always easy! UMKC Conference on the Stock Flow Consistent Approach. UMKC, UMKC.
Li, T.-Y. and J. A. Yorke (1975). "Period Three Implies Chaos." The American Mathematical Monthly **82**(10): 985-992.
Lorenz, E. N. (1963). "Deterministic Nonperiodic Flow." Journal of the Atmospheric Sciences **20**(2): 130-141.

Lux, T. (2001). "Turbulence in Financial Markets: The Surprising Explanatory Power of Simple Cascade Models." Quantitative Finance **1**(6): 632-640.
McManus, M. (1963). "Notes on Jorgenson's Model." The Review of Economic Studies **30**(2): 141-147.
Minsky, H. P. (1982). Can "it" happen again? : essays on instability and finance. Armonk, N.Y., M.E. Sharpe.
Poincaré, H. (1905, 1956). "Principles of Mathematical Physics." The Scientific Monthly **82**(4): 165-175.
Prescott, E. C. (1999). "Some Observations on the Great Depression." Federal Reserve Bank of Minneapolis Quarterly Review **23**(1): 25-31.
Ramsey, F. P. (1928). "A Mathematical Theory of Saving." The Economic Journal **38**(152): 543-559.
Solow, R. M. (2006). Reflections on the Survey. The Making of an Economist Redux. D. Colander. Princeton, Princeton University Press.
Sonnenschein, H. (1972). "Market Excess Demand Functions." Econometrica **40**(3): 549-563.
Sushko, I., M. Wegener, F. Westerhoff and G. Zaklan (2009). "Endogenous Business Cycle Dynamics within Metzler's Inventory Model: Adding an Inventory Floor." Nonlinear Dynamics, Psychology, and Life Sciences **13**(2): 223-233.
Varoufakis, Y. and C. Arnsperger (2006). "What Is Neoclassical Economics? The three axioms responsible for its theoretical oeuvre, practical irrelevance and, thus, discursive power." post-autistic economics review **38**.
Velupillai, K. (1998). "Richard M. Goodwin 1913-1996." Economic Journal **108**(450): 1436-1449.