Introduction to Mathematics

Introduction
I’d like to tell you a story today – the story of my day. This morning my alarm sounded at 6:15 am; I promptly hit the snooze button, which allowed the passage of another ten minutes before the alarm sounded again. When I got up, I turned up the thermostat so that the temperature inside the house increased from 55 degrees to 65 degrees.

I went to the kitchen to start the coffee. I poured 12 cups of water into the machine, but only 10 cups of coffee actually appeared in the decanter. I idly wondered where the rest of the water had gone …

The coffee started, I went outside to get the paper and noticed how cold it was outside. Must be below freezing, I thought, and today feels even colder than yesterday.

At breakfast, we had bacon. Evan, the self-proclaimed bacon monitor, says we each get 2 ¼ pieces. Dave says two are enough for him, so Evan adjusts and now the rest of us get 2 1/3 pieces.

The phone rings three times before we can get to it. Gramma had entered the series of numbers that connects her phone to ours – she also remembered that her time is three hours ahead of us; while it is only 7:00 am in Portland, it is three hours later in Florida.

Evan is deciding whether he should leave now and catch the 7:58 bus, or if he should wait and take the 8:10 bus. He takes the 70 bus and likes to wait for a friend who takes the 14 and meets him at the bus stop on 11th Ave.

I get in my car to drive to work, and do a quick check of the dashboard. I shift from first to second gear when the RPMs are around 3, which is actually multiplied by 1000 to determine how many revolutions per minute. I don’t care too much about that, however, I just want to keep my speed at 25 miles per hour. I am momentarily distracted when a light comes on and reminds me that I only have 3 gallons of gas left in my tank. I estimate I have enough time to stop at the gas station without being late to work.
I check the calendar, and note the day, the week, and the month, and am reminded that Julian is rehearsing with his accompanist this afternoon. The last time they practiced, she asked him to work on the second section with a metronome so that he could accurately subdivide the quarter notes so that exactly four 16\(^{th}\) notes fit into one beat. Before they even begin the rehearsal, she will play an A on the piano, and he will play his A- he will adjust the tension on his violin string so it vibrates at exactly A 440Hz.

After the rehearsal, which lasts about an hour, I will write her a check for $45. When she cashes the check, that amount of money will be deducted from my checking account.\(^1\)

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Our lives are flooded with numbers. Just think of all of ways we use numbers in the brief example I just illustrated: time, temperature, estimation, fractions, subtraction, addition, multiplication, division, measurement, frequency waves, volume, money, phone numbers, bus lines, street names, music notation, and monitoring the performance of our automobiles.

Even the words I used reflect mathematical concepts: promptly, increased, more, later, enough, ahead, now, figure, deduct, and amount. Mathematics is a part of our human culture; we use mathematical concepts every day to describe and represent our world. It is essential for children to understand the mathematical concepts behind these words and numbers if they are to become fully adapted to the culture we live in today.

Mathematics and Cultural Adaptation
Mathematics is part of our human heritage, organically growing out of normal development. In its broadest form, mathematics is about patterns. Our human brains are designed to recognize and understand patterns, and our ability to manipulate patterns and communicate about them is a product of humanity, perhaps more important today than ever before. We are born with an innate number sense. We possess a mind that is mathematical in nature, and we have inherited a broad potential to fully develop mathematics. However, our mathematical success, attitudes, and ultimately, our mathematical abilities are shaped by the experiences in our environment.

\(^1\) Brian Butterworth inspired this everyday account of how we “use numbers every waking hour of every day,” as he read mathematical references in the morning paper in the preface of his book, *What Counts – How Every Brain is Hardwired for Math.*
As we explore the acquisition of mathematical ability, you will find many interesting parallels to the acquisition of language. We will use these parallels to demystify the development of math and bring us to the realization that mathematical ability is a natural human development and as such, is perfectly suited for the child in the first plane.

You will see why and how we offer mathematics in the Casa, taking advantage of the sensitive periods, the human tendencies, and the first plane child’s inherent capacity to love all that is in her environment. Every child has the potential to embrace mathematics with confidence, ease, and delight, because it is a natural part of her cultural heritage.

Stanislas Dehaene, a mathematician and cognitive neuropsychologist writes, “I am certain that children of equal initial abilities may become excellent or hopeless at mathematics depending on their love or hatred of the subject. Passion breeds talent – and parents and teachers therefore have a considerable responsibility in developing their children's positive or negative feelings towards mathematics.” (Dehaene, The Number Sense, How the Brain Creates Mathematics, p. 9)

As Montessori educators, we recognize the potential for mathematical development in the lives of young children. Like every other aspect of child development, we can either support, or thwart it. Ultimately, every experience the child has with mathematics becomes a part of the biological makeup of the child’s brain. The neural connections and chemical transmissions within the child’s brain reflect the child’s level of interest and emotional involvement (Dehaene, 1997, p.8). Through our Montessori math presentations we support child development by aligning cognitive growth with our educational practice. In this way, we create the best possible experience with mathematics.

What is Mathematics?
As we’ve said, math is about patterns. Math is the science, or study, of patterns, and patterns are in every aspect of life- physical, biological, and sociological. There are patterns in the movements of the sun and moon, the tides, and the winds. There are patterns in the growth of plants, the design of a spider’s web, and in the migrations of birds and butterflies. There are patterns in the complex relationships of people, patterns in our speech, and patterns in our behaviors.
The types of patterns we study create the different branches of mathematics. For example, arithmetic is the study of the patterns in counting and quantity. Algebra uses symbolic notation to describe the generalized patterns in arithmetic. Geometry studies patterns of shape. Calculus and physics study the patterns of motion, change, and space; probability studies patterns of chance, and logic studies patterns of reasoning.

As our lives and technological innovation become more complex, so does mathematical thinking. Around the turn of the century, in the 1900s, mathematics consisted of only about twelve subjects: arithmetic, geometry, algebra, calculus, and so on. But today there are between 60 and 70 different categories of mathematics. There has been an “explosion in knowledge” in the past century, as known branches of mathematics split up into subfields and entirely new theories of mathematics arise (Devlin, 2000).

Despite the extensive branching of modern mathematics, it is by examining the roots of mathematical understanding that we understand the importance of mathematics in the early years of life. Mathematical ideas are not radically different than other more commonplace notions; our brains are designed to create abstractions and manipulate them. What better example of this ability than the power of very young children to create and use language.

Language Development and Mathematics
To help us understand the nature of mathematics and why we consider it an accessible subject for children under age six, let’s look at what we already know about children and language development.

We defined language as “a socially shared code or conventional system for representing concepts with arbitrary symbols and the rules governing the combinations of those symbols.” Following this definition, isn’t mathematics also a socially shared code or system that represents concepts with arbitrary symbols? Doesn’t mathematics also involve the rules governing the combinations of those symbols? It would seem that math and language are two expressions of the same basic process.

Like language, mathematics requires concrete sensorial experiences, creating abstractions based on those experiences, and understanding the structures and rules to using them. Let’s summarize

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2 This definition of language (Owens, 1992) is used in my Introduction to Language lecture.
what we know about language development. By the age of three or four, children are generally fluent in their native language, and many children are fluent in more than one language. They begin with non-verbal communication, explore with the basic sounds in their environment, and begin to combine those sounds into words. In acquiring vocabulary, children attach verbal symbols (words) to abstractions formed from concrete experiences in their environment. Single words become two word phrases, and before long, children have mastered the phrase structure of language and can apply rules governing the structure. They manipulate abstractions by plugging in specific words into the structures to convey intended meaning. They internalize the patterns of behavior and meaning that add nuance and inflection to their words.

We know that children have an innate capacity to acquire language. We agree every child has the potential to create abstractions and apply the patterns, structures, and sequence of the language that surrounds her. This sounds very complex, and indeed it is, but no one thinks language learning is too abstract for young children. We expect it, and we support it, because we know that language development is dependent upon the language used in the child’s environment.

However, as Mario Montessori wondered, “If, because of the abstract nature of language, people succeeded in protecting the child’s mind from it until he was ten years of age, I wonder what kind of language the child would speak and whether he would not find language as distasteful a subject as the majority of children seem to find mathematics.” (Mario Montessori, (1961) “Maria Montessori’s Contribution to the Cultivation of the Mathematical Mind,” NAMTA Journal, Vol.19, No.1, Winter, 1994)

The cognitive ability that evolved in the human brain making language possible is also the ability necessary for mathematical thinking. This ability is pattern recognition. Our brains have developed to recognize patterns. We recognize visual patterns, linguist patterns, social patterns, and many others. “Mathematical thinking, as practiced today, makes use of mental capacities that were developed hundreds of thousands, and in some cases, millions of years ago. Doing mathematics does not require new mental abilities, but rather a novel use of some existing abilities.” (Keith Devlin, The Math Gene, p.180)

The brain’s existing ability to recognize patterns and react accordingly is what underlies our ability to develop and use language, and also to develop and use mathematics. It is our “mathematical mind” at work, constructing itself out of the patterns in our environment. It is the mathematical mind.

Further exploration of this topic found in Chapters 6 and 7: “Born to Speak,” and “The Brain That Grew and Learned to Talk” in Devlin’s book The Math Gene.
mind that made it possible for both language and mathematics to develop from an evolutionary standpoint, and it is the mathematical mind that enables the individual child to develop language and mathematics.

Just as we recognize a specific universal pattern to how children develop language, there is also a universal pattern to how children develop mathematics. Understanding the basic history of how our human species developed mathematics is important to our work with young children, because within each individual child we can see a recapitulation of the developmental process of our entire human species. Recognizing this developmental process clarifies the natural sequence of concepts that bring a child to mathematical comprehension.

A Brief History of Mathematics

View an entertaining and useful short film on the history of mathematics at:
http://www.youtube.com/watch?v=cy-8IPVKLo

“Ontogeny Recapitulates Phylogeny,” or, “The Development of Mathematics”

Our history indicates that when human beings first began to use language, they used only words for 1, 2 and possibly 3. This may be because “oneness, twoness, and threeness are perceptual qualities that our brain computes effortlessly, without counting” (Dehaene, 1997, p. 92).

Numerous studies show that newborns only a few days old can detect a difference between two and three objects. This ability to recognize quantities of small numbers is called “subitizing.” These various studies, replicated in different countries and with different controls, show that regardless of variables in any physical parameters - shape, color, size, and even moving displays - babies appear to notice the constancy of objects and extract their numerosity. (Dehaene, 1997, p. 50).

Additional studies have also shown how babies can intuitively add and subtract small numbers. This is not a conscious act but a perception and expectation. The babies indicated that when they were shown one object plus another object, they expected to see two objects (Dehaene, 1997, p.54; Butterworth, 1999, p.107). This ability is also found in animals, indicating that basic number sense or perception has an evolutionary place deep within animal and human brains.

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4 E.H. Haeckel (1834-1919), a German biologist and philosopher, theorizes that the embryological development of the individual repeats the stages in the evolutionary development of the species.
In addition to this innate number sense, early mathematical ability also includes neural connections formed through \textit{sensory experiences} with spatial relations, vertical orientation, cause and effect, gravity, physical force and coordinated movement. Our human ancestors must have had daily experience with these concepts as they built shelters that would withstand the elements, experimented with more effective ways to hunt animals and gather food, and cared for their basic needs – all without a formalized system of numbers. We see this exploration and growth in the baby who is reaching out to grasp an object, dropping a toy repeatedly on the floor, learning to walk upright, and attempting to climb up the stairs.

The next basic mathematical development is \textit{one-to-one correspondence}. Some of our oldest records of this are the calendar type markings found in the caves of Lascaux, France dating back to around 18,000 BC. These drawings show similar markings for each day and then a specialized marking- possibly indicating a phase of the moon. Historians have theorized that similar markings might be tracking the movements of the moon, seasonal changes, or reproductive cycles. We see this same type of one-to-one correspondence in very young children as well – one sock goes on one foot, one arm goes in one sleeve, or when a two year old sets the table and puts out one fork for mama, one for daddy, and one for bubbly.

\textit{Even though this same child may be able to count by rote-} “one, two, three,” he cannot count out a group of three forks - that is an abstract connection he has not yet made. The child setting the table is only setting out a collection of “ones”- one fork here, one fork here, one fork here- one, one, one, much like the individual tally marks that our ancestors made to mark the passing of one day.

The idea of one-to-one correspondence symbolized by tally marks gradually evolved to the use of everyday objects such as clay markers, shells, or beads to \textit{represent quantities}. The Sumerians filled spheres of clay with marbles to represent the objects they counted, the Incas tied knots on strings, and the Romans used vertical lines or bars for their first three digits. Interestingly enough, the word “calculate” comes from the Latin word for “pebble” reflecting the time when numbers were calculated by moving pebbles on an abacus (Dehaene, 1997, p. 97).

It was the limitations of one-to-one markings that led to the use of symbols representing groups of objects. This reflects a level of symbolic thinking that led to our current understanding of a number word that \textit{represents a group of objects as well as an individual quantity}. Now we see the little child who knows that there are four people eating dinner and so can count out four forks from the silverware drawer and bring them to the table. He understands that the symbolic word
“four” means a collection of four objects, and that four is more than three. The concept of associating a physical quantity with a symbol and the corresponding number word is the foundation for all mathematical understanding.

The next historic and developmental milestone comes from the concept of zero. Connecting the concept of zero to the numerical symbols brings us to the place-value system. There is evidence that the Babylonians, Chinese, Mayan, and Indian civilizations all developed the use of zero as a placeholder, but the idea of zero as a “null quantity” probably came later. With the use of place value, the manipulation of quantities, what we call “arithmetic,” became much easier. With zero as a placeholder, we don’t need to count each object to find a quantity; we can manipulate numeric symbols whose value depends on the place they occupy. It is much more efficient for the human brain to recognize a symbol and associate it with a quantity than it is to count a collection of objects.

Once the child understands our system of place value, the number 4 can represent four ones, 40, 400, 4000, and so on, making manipulation of larger quantities quite manageable. Now the child can approach the patterns of manipulating quantity that we call arithmetic: addition, subtraction, multiplication, and division. This understanding opens the gateway to other patterns of numbers: those less than one, as in fractions or decimals; generalizing the patterns by representing them in variables, as in algebra; and connection the patterns of shape with number and letter as we see in geometry.

Within the individual child we see a reflection of the history of the development of mathematics. An innate, unconscious number sense underlies sensory experiences in the environment, leading to the formation of abstractions. An understanding of one-to-one correspondence evolves to an understanding of quantity as a group or set of objects, then the association of verbal and numerical symbols that represent quantities. From there, the child develops an understanding of the power of zero and how place value affects quantity. Finally, she discovers the patterns of how those quantities can be manipulated in the form of arithmetic.

This mathematical foundation becomes the engine that drives all later exploration. The child constructs his own toolbox of mathematical understanding that can be used to create any number of imagined constructions.
How is Mathematics Presented in the Casa?

The obvious question at this point is, “If mathematics is such a natural, organic process in human beings, why do so many people find mathematics so difficult or unappealing?” The answer cannot be that math is too abstract, or unnatural, or only for special, gifted individuals, because all humans are born with an innate numerosity, a mathematical mind uniquely capable of recognizing patterns, and an ability to create and use abstractions as evidenced by universal language development.

I think the answer is found in how mathematics is offered. We know that children learn from concrete experiences in their environment. Children create abstractions from these concrete sensorial experiences, and then attach language to those experiences to fix the abstractions. Experience precedes language. Once the experiences have been labeled with language, the abstractions can be manipulated purely in the mind.

However, when people generally introduce math to children, they begin with the abstraction, rather than the concrete experience. They start with the numerals, showing the symbol and naming it- “this is one, this is two…” and so on. Sometimes they might show two apples, another time “two” is on the calendar, still another time, “two” is how old someone is. This is often confusing for the child, and creates an unconscious barrier to numbers that manifests in resistance or lack of interest. (Mario Montessori, London lecture #25) This approach to learning mathematics could be compared to expecting a child to learn to talk by teaching her the alphabet.

Therefore, an effective approach to mathematics would be to offer a concrete sensorial experience of the concept, and give time for experimental interaction so each individual child can form her own abstractions. “In the growth of symbolic/abstract reasoning, the concrete building block is the foundation. Seeing and touching come first (three real cookies) then understanding that things (pictures, words, or numerals) can stand for other things (a “3” for the idea of three cookies). Only after these stages are mastered through many individual experiences and the accompanying brain maturation occurs can children start to move to abstract thought.” (Jane Healy, Your Child’s Growing Mind, p. 332, 1987)

This is, in fact, exactly what Montessori offers in the mathematics area of the Casa. Montessori understood “that the teaching of arithmetic should be completely transformed. It should start with sense perceptions and be based on a knowledge of concrete objects.” (Montessori, The Discovery of the Child, “Further Developments in Arithmetic,” Chapter 19, p. 278, Ballantine)
When children or adults have resistance or barriers to mathematics, it is not a question of not having mathematical ability, but of not having made an abstraction. An abstraction cannot be taught unless an experience is already there. The child has to create a body of unconscious experiences with mathematics before she can construct an abstraction. (Mario Montessori, London lecture #25) This is the role of indirect preparation for mathematics.

Indirect Preparation for Mathematics in Practical Life, Sensorial, and Language

The child’s mind is prepared for mathematics so that when she is presented with the first activities in the math area, she will understand and be keenly interested. The mind must have a level of preparation in order to understand. Montessori uses the example that if I spoke to you in Spanish, you would not be able to comprehend what I was saying, no matter how many times I repeated it. Your mind would not be prepared to listen to Spanish. However, if you learned a few words of Spanish, as you listened to me explain something, your ears would perk up at the words you understood. You would have enough to pique your curiosity and instead of the words being incomprehensible, you’d want to learn more.

Another way to think of this mental preparation is that if one person came up to you and told you some new information, you might not understand or remember it. If a second person told you the same information, the ideas would be a bit more familiar and stay with you a while longer. By the time a third person came to tell you, you would already have a good idea of what he was going to say, and are fully prepared to hear the full story. This is how the mind organizes the intelligence based upon experience; one experience relates to another, forming a subconscious framework of intelligence.⁵

Practical Life

Some of the most important concrete preparations for mathematics happen before the child even approaches the math shelf in the classroom. Within the activities of practical life are essential first-hand experiences with cause and effect, logical sequence, spatial relations, vertical orientation, gravity, physical force, and coordinated movement. Every time a child gauges the amount of water needed to fill a pitcher she experiences volume. When she exerts just the right amount of pressure to rub a shine to a brass object, it is a lesson in physics. When she lays out the objects of an activity in the order of use, she explores logical sequence. As she endeavors to walk across the room carrying a heavy pitcher of water without spilling, she is gaining experience and mastery in spatial relationships and moving her body through space.

⁵ These two examples are found in Creative Development 2, Chapter 2.
Mathematics is supported in the daily life of the classroom as children help prepare just enough snack for everyone, when they practice the social patterns of behavior in the lessons of grace and courtesy, when they set the tables for the children who eat lunch at school, when they move to the rhythm of the music with activities on the line, and when they measure one scoop of food for the guinea pig. The children are surrounded every day with opportunities for concrete experiences that support mathematical thinking.

Life itself provides feedback in terms of these experiences. The materials and activities help the child assess her own abilities. If too much water is poured, it spills out. If she moves too quickly or misjudges the amount of space, she bumps into something. If she puts the soap on a dry table, working out of sequence, suds won’t form. This feedback comes in a neutral, nonjudgmental way, and the child has the opportunity to adjust or make changes herself in order to be more successful.

Sensorial
While working with the sensorial materials, children refine the sensory perceptions necessary to discriminate fine differences between objects. They have multiple opportunities to practice matching, finding the one-to-one correspondence between two objects. They make sets of objects sharing the same characteristics, and discover that some objects are members of more than one set. The children discover the relationships between a series of objects as they grade them by increasing or decreasing intensity or size. The activity of grading, “showing understanding that a block, or number, can be bigger than one neighbor and smaller than the other – all at the same time... is prerequisite to a true understanding of counting.” (Jane Healy, Your Child’s Growing Mind, p. 325, 1987)

All the while, children are classifying and categorizing the information taken in through their senses, and creating neural connections for logical, orderly thinking. The self-correcting nature of the activities invites concentration and repetition, leading to increasing levels of self-perfection and understanding. The sensorial materials isolate individual qualities, and are designed and presented with exactness and precision, so the abstractions that the children form are clear and accurate. The language we give to name those abstractions gives the children the opportunity to manipulate their abstractions solely in the mind.
In order to be successful in mathematics, it is necessary for children to develop two basic abilities that are fundamental to arithmetic, algebra, geometry, statistics, trigonometry and calculus.

1) First, and most important, is the ability to comprehend relationships. Called “spatial-temporal thinking,” it is the ability to mentally manipulate relationships in time and space.

2) The second is the ability to follow rules, observe accurately, and follow an orderly line in problem solving. This relates to an orderly mind and keen observational skills. (Healy, Chapter 11, 1987)

The hands-on experiences in practical life and sensorial support both of these key abilities: comprehending relationships, and developing an orderly line of thinking. However, the child’s continued exploration of spoken language also provides important indirect preparation for mathematics. Spoken language involves the communication of ideas and information by using verbal symbols (words).

**Spoken Language**

But there is something else important about language. Language includes particular words whose purpose aids the manipulation of abstractions. Words like, *unless, because, if, then, and, or, and every* change the meaning of the utterance; they are used to manipulate the abstraction, adding syntax, or grammatical structure. Using these words changes simple vocabulary into more complex expressions. For example, I could ask, “Do you want a cookie?” This is a very simple expression- it could even be reduced to “Cookie?” Or, I could ask, “Do you want a cookie? *Because, if* you do, *then* we’ll have to bake some, *unless* your brother didn’t eat every one, *or*, we made more than I thought.”

When little children move from simple vocabulary to more complex sentence structure, they are doing more than just expressing themselves; they are actually changing the language system into something very different. The simple words can be used in very different circumstances, for very different purposes. (Devlin, *The Math Gene*, p.148)

Think about how this relates to the process of applying operational symbols (structure) to numerals (vocabulary): 4+5 is very different than 4x5; 4/5 is very different than either 4 or 5. This is like the children moving from simple vocabulary to more complex sentence structure. Ex: “Cookie?” becomes, Cookie *and* milk? Cookie *or* milk? Cookie *before* milk? Cookie *after* milk?
Here is another example. Think about this sentence: “Whoever did this, must be in a hurry.” Here, the pronoun structure is the same basic structure as the algebraic formula “If $x + 2 = 7$, then $x = 5$”. $\text{Whoever (x,) did this (+2=7) must be in a hurry (then x=5).}$ The same basic structural understanding can apply to linguistic and mathematical thinking.  

You can see how understanding the patterns or structures of spoken language can provide a prerequisite for understanding the patterns and structures of mathematics. With this preparation in place, the child need only form the “new abstractions” of quantity, learn the “new vocabulary” of the numerals, and then sit back and enjoy discovering how these concepts can be manipulated. “**Mathematical material in particular, presented in the sensitive period in suitable fashion, permits the child to understand fundamental truths, and not only that, but to discover new relationships.** “It is then that the [child’s mind] can give surprising revelations. The child becomes indefatigable in work, provided we offer the necessary means.” (Montessori, “The Psychology of Mathematics,” an address given to the Cambridge Education Society at Trinity College on October 16, 1935, reprinted in AMI Communications, 1971)

### The Organization of the Montessori Math Materials

The materials in the math area of the Casa are organized into six groups:

- The Numbers One to Ten
- Continuation of Counting
- The Decimal System
- The Memorization Exercises
- The Passage to Abstraction
- Fractions

These groups of activities can be thought of as many limbs of a tree, all branching out from a strong, sturdy trunk, which is the understanding of the numbers one to ten. The continuation of counting, the activities with the decimal system, and the memorization exercises have many parallels, and then these lines converge again with the passage to abstraction, followed by the fractions.

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6 This example comes from John Allen Paulos’s review of the book, *Where Mathematics Comes From*, written by George Lakoff and Rafael Nunez.
There is also a general pattern of activity to each of the six areas:

- Quantity in isolation
- Symbol in isolation
- Association of quantity and symbol
- Repetition, with variations
- The “test”

First, the quantity is introduced with concrete sensorial materials. When the child shows understanding of the concept of quantity, we present the numeric symbols representing those quantities. When the child understands the symbols, there are activities and games for associating the quantity and symbol together. Next there is repetition with variety or extensions, and finally, there is a “test” - an activity or game designed to see if the child has generalized the concept.

There are several general principles applied to the presentation of math materials.

- All new concepts are introduced with concrete, sensorial materials.
- Only one new concept introduced at a time.
- Three period lessons are used to teach new vocabulary.
- We present the whole of an idea or area first, and then examine the parts in more detail.
- Each concept builds on ideas previously learned, gradually adding more and more detail or information.
- The emphasis is on exploration and discovery.
- Each child is given whatever time needed to make her own discoveries and build her own abstractions.

At every step of the way, games, mini-challenges, designs, and patterns insure that math is fun, relevant, and accessible. “Children are only too pleased to learn mathematics if only one shows them the playful aspects before the abstract symbolism.” (Dehaene, p.143, 1987)

Montessori understood the importance of the early work with mathematics when she wrote, “The seed of mathematics must be very, very carefully sown. We must not confuse the trunk of the tree with the branches. We must certainly not expect good branches to grow on a dead tree. Many people reduce mathematics to certain feats of memory. If we make the child learn that three and three make six, or two times four make eight, we are constructing a tree, by nailing small dead pieces of wood to a larger piece of dead wood so that it resembles the trunk of a tree with branches attached. In doing so, we have only created an illusion. It is no tree, only a cross. It will bear no fruit. It will give no life or joy, only suffering. Instead
"if we plant the seed carefully, we can watch the little plant take firm root, sprout leaves, and grow strong branches with pleasure." (Montessori, *Creative Development of the Child, Vol.2*, p. 23)

**The Numbers One to Ten**

Although children between the ages of 3-4 (and sometimes younger) know how to count, they often do not know why to count. Adults know that the purpose of counting is to determine the number of items in a set, and that the final number counted is the one that matters- it is the cardinal number of the set. Young children, although they have mastered the mechanics of counting, do not seem to understand that the reason for counting is to find out how many are in the group.

This is why we do not present the number rods to three year olds, but wait until age 4, or even a little later. Around age 4, the basic internal number sense converges with the oral language of rote counting, bringing meaning to counting. At first, he may be surprised to find that the last number he counted tells “how many,” but after several repetitions, he will be able to consistently infer that counting will tell him how many. (Dehaene, 1997) (Montessori, *Discovery of the Child*, “Teaching How to Count and an Introduction to Arithmetic” pp.263-264, Ballantine, 1967)

To assist the development of “how many,” we begin with the number rods. The number rods correspond to the proportions of the red rods, which the child has explored in the sensorial area. But whereas the child estimated length with the red rods, he can count and calculate exactly with the number rods. The unit of measure in the red rods becomes the actual unit of quantity in the number rods. In this way, we help the child build the abstraction of quantity onto his abstraction of length.

The number rods begin by counting the sections of each rod, and saying “how many.” We offer lots of practice and games with counting and naming the different quantities of 1 to 10. Only after the child is comfortable with the purpose of counting- to name the quantities, do we introduce the numerical symbol for each quantity.

The symbols are introduced as sandpaper numerals, with three period lessons in the same manner as the sandpaper letters. When the child is able to consistently identify the numerical symbols, we offer several activities in associating the quantity and the symbol. Other activities with the
numbers 1-10 introduce the ideas of combining or decreasing quantities, “sets” of numbers, zero as “the empty set,” odd and even numbers, and finally, there is a little “test,” a game that shows whether the child can generalize the quantities and symbols 1-10 to any objects in the environment.⁷

The concepts introduced with the numbers 1-10 are the foundation for all work with mathematics; they form the trunk of the tree. Only with a secure, solid understanding of these concepts can the child proceed onwards toward other aspects of mathematics. It is important that every child be given all the time she needs to understand for herself the concepts contained herein.

Once the work with the numbers 1-10 are secure, the child can branch out and explore the activities in the decimal system, the continuation of counting, and the memorization work. There are many parallels activities in these three areas, and they should not be thought of as linear, but as many different expressions of what can be done with the numbers 1-9 and the concept of 0.

**The Decimal System**

Children love large numbers. With the use of zero, we can introduce the concept of place value, and create very large numbers for the children to explore. We introduce units, tens, hundreds, and thousands following the general pattern of concrete quantity, then symbol, and then associating the symbol with the quantity. The games that follow show what can be done with these large quantities: we can put them together, we can take them away, we can put the same quantity together many times, and we can share out a quantity equally. These games provide the concrete experience with the operations of addition, subtraction, multiplication, and division, but the children discover the nuances themselves.

The root of the decimal system is the interplay between nine and ten. The children explore this characteristic of a base 10 number system when they discover we can only have up to nine of any one category before it is transformed into one of the next higher category. The children also discover that this process can work in the other direction too- if I need more of a smaller category, I can borrow from a larger one. The playful nature of these various games invites a great deal of repetition. Over and over the children experience the processes of carrying, borrowing, changing from one category to the next, and all the while developing a very solid, concrete understanding of what is occurring during the operations of arithmetic.

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⁷ Montessori describes the activities with the numbers 1-10 in chapter 3 of Creative Development of the Child, Vol.2; and chapter 18 in Discovery of the Child.
Continuation of Counting – “Teens and Tens”
Parallel to the activities with the decimal system is the exploration of consecutive numbers greater than ten. The presentations in the continuation of counting follow the same pattern of activity the child has experienced before- quantity, symbol, association of the two, activities inviting repetition with variety [100 and 1000 chains], and a built in assessment, or “test” to insure understanding and generalization of the concepts. [Note: The application of counting to the square and cube chains of 1-9 could be considered a “test’ although less of one than the memory game with numbers or word problems.]

With these activities, the children discover that ten and any number of units creates the group of numbers we call the “teens,” and that any number of tens and any number of units create the next body of quantity we call the “tens.” Following work with the teens and tens, the child is ready to explore the counting chains, beginning with the square and cube of ten, counting to 100, and then all the way to 1000. The patterns of counting are reinforced with the introduction of the square and cube chains of 1-9 as well. The design and activity with the counting chains indirectly prepare the child for multiplication, and later work with squaring and cubing numbers.

The Memory Work
Another avenue of exploration that takes place with the numbers 1-10 involves the essential combinations that we commit to memory. Montessori advised that we avoid the term “memorization,” and instead, speak to the children about discovering that they know the basic combinations by heart, because she recognized the connection between love and interest. Committing the essential facts of every operation to memory gives us ready access to more complex mathematical structures, much like learning the sounds of the alphabet brings us to reading and understanding the written words of any author.

Building on the concepts of the operations previewed with the number rods, and introduced concretely with the decimal system beads, children explore these combinations using beads and number strips, gradually moving towards more symbolic materials until they have learned them by heart. We take advantage of the power of the absorbent mind to learn the math facts easily and without conscious effort, while the child is still in the first plane of development.

The memory work also follows the general pattern of activity of all math materials, focusing on discovery, exploration, and repetition with variety. It is the child herself who discovers with

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8 Montessori, in the pamphlet “The Decimal System,” published by AMI.
surprise and pleasure that she already knows that 7+3=10, and guess what? 2+8=10, and 5+5=10 too! These discoveries belong to the child; the Montessori materials and presentation simply provide her with the means to make those discoveries.

**Passage to Abstraction**
After the parallel activities of the decimal system, the continuation of counting, and the memorization work, all of the abilities and concepts explored and mastered in terms of place value, the operations, and the essential combinations of math facts converge with the group of activities we call the “Passage to Abstraction.” It is here, as the child works with the most symbolic materials, that she realizes she can find the answers quicker in her head than by manipulating the materials. During this passage to abstraction, the child begins the mental transition from the first plane where she manipulated concrete objects to the second plane, where she manipulates abstractions.

**Fractions**
The final group of activities gives the child a taste of all that is available to her in the further exploration of mathematics. The work with fractions shows the child that she can apply the abilities and concepts she has mastered to another branch of numbers: those smaller than one unit. Although the principles are much the same as with whole numbers, conceptually, this requires more brain maturation and for this reason, is presented to the child after all of the other mathematics work in the Casa.

**The Developmental Purposes of Mathematics**
By this point, I hope you consider mathematics a natural development in human beings and an integral part of the child’s cultural adaptation. But why offer mathematics to children under the age of six? We include mathematics in our work with young children because mathematical exploration supports child development.

Remember, Dr. Montessori did not set out trying to teach the children mathematics; her interest was in the child himself. She was trying to discover the process of natural child development, and the various activities of the Casa evolved as vehicles for developing particular natural processes.

With her realization that every human being possesses a mathematical mind came the conclusion that we could support the development of the child’s mind and integration of personality by
supporting the mathematical mind. It is a mind that constructs itself from organizing and ordering the patterns it finds in every aspect of life.

Through the activities in mathematics, the mind and the hand are working together to construct the intelligence. The materials themselves are a physical representation of mathematical thinking. They appeal to the senses and they involve manipulation by the hands in order to extract the essence of the mathematical concept being presented. The mind creates the abstraction from the experience of the hands. The math materials are also presented in relation to each other, and in the context of the environment. Montessori identified an essential principle in education when she wrote, “to teach details is to bring confusion; to establish the relationship between things is to bring knowledge.” (Montessori, From Childhood to Adolescence, p. 58, Clio)

Mathematical exploration is a vehicle for the human tendencies to operate. When we see evidence of the human tendencies, we know that the intelligence is engaged, the mind is working, and the individual is satisfied. This is as true for you today as it is for children under the age of six. Think about what we have been examining during the course of this lecture, and the human tendencies: order… exploration… communication… work/activity… exactness… self-perfection… abstraction… imagination… The human tendencies are urges, or drives, that remain with us throughout our lives. As connected as the human tendencies are with the mathematical mind, we cannot serve the child’s development if we do not understand and nurture the development of the mind and support the human tendencies.

Each of the sensitive periods also has a role in the development of the mathematical mind and in creating indirect preparation for mathematical thinking. The child under age six is driven by the sensitive periods for order, movement, refinement of sensory perceptions, and language to seek out that in her environment that will support her development.

The child under age six is in the process of creating the adult he will become. This is the time of the conscious absorbent mind, and he will take from the environment all that his culture has to offer. If mathematics are a part of this environment, then the child will accept and love mathematics in the same manner as he accepts and loves the music, food, climate, attitudes, and people of his culture. Mathematics will become a part of him, because cultural adaptation is a process of becoming and belonging, from both a spiritual and a biological perspective.

For development of these ideas, read Kay Baker’s excellent article, “The Mathematical Intelligence Seen Through the Lens of the Montessori Theory of the Human Tendencies,” NAMTA, Spring 1996.
Conclusion

I plan to leave Montessori Northwest at 4:30 today. On the way home, I'll stop at the store and pick up a nice three-pound chicken for dinner- they usually cost 2.99/pound, so I'll pay about $9 for our dinner tonight.

Evan has basketball practice from 5:00 – 7:00. He has outgrown his shoes, size 8, and will soon need a new pair in the next higher size. Hopefully, we can find a pair for less than $50! There are 11 children on his basketball team, so there are enough for two teams of 5, with one person always rotating out. When they play scrimmages, each player matches up with one player on the other team, and both teams attempt to score 2 points by making a basket. Each player gets 5 fouls, and if a foul is committed, the other team gets the opportunity to score one or two points, depending on the situation.

While Evan is at practice, I'll start dinner. I'll roast the chicken at 350 degrees for about an hour and a half – give or take 15 minutes. I think a little rice pilaf will be nice, so I measure one cup of rice and about two cups of broth- it will take about ½ an hour to cook.

While I'm preparing dinner, perhaps I'll put on a little music. I think I'll listen to the first movement of Brahms's double concerto for violin and cello, opus 102. I'll muse on the fact that an opus is numbered work, and wonder how many pieces of music Brahms wrote in his life. He was born in 1833 and died in 1897, so that means he lived for 64 years...

Once again, we see how mathematics permeates the fiber of our daily lives. Time, weight, temperature, size, measurement, scoring, ratio and proportion, life and death... there is no question that our lives are filled with mathematics. The child's life too, begins and ends with mathematics. At birth, we see an infant, in possession of a mathematical mind, who grows and develops into a human being capable of creating and imagining that which has never existed before.

Montessori said in the Absorbent Mind, "If we study the works of all who have left their marks on the world in the form of inventions useful to mankind, we see that the starting point was always something orderly and exact in their minds, and that this was what enabled them to create something new. Even in the imaginative worlds of poetry and music, there is a basic order so exact as to be called "metrical" or measured." (Montessori, The Absorbent Mind, p.185, Kalakshetra)
Later in that chapter, “he does not at first absorb the actual mental riches of his [culture] but only the patterns which result from them. He absorbs... the basic or summarized part, which is repeated in the habitual life of the people. He absorbs, in short, the mathematical part. Once the patterns have become established within him, they remain as fixed characters... Later on, a man may develop himself indefinitely, but it will always be his foundation.” (Montessori, The Absorbent Mind, p.189, Kalakshetra)

Montessori too was amazed at what the children were able to accomplish in mathematics. She tells us, “But what was really wonderful, was to see the great spiritual happiness of the children, their amazing enthusiasm, their persistence with self-imposed, difficult problems, and their great joy whenever they arrived at the solution, that was often reached through channels unheard of by the teacher herself.” (Montessori, “Psychogeometry and Psychoarithmetic,” AMI Communications, no.1/2, 1982)
References
Montessori, Dr. Maria, (mid-30s). Psychogometry and psychoarithmetic. *AMI Communications*, no.1/2, 1982.

Other Sources